III. 1/N Expansion for U=∞

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- For U=∞ $\Rightarrow$ no "doublons" $\Rightarrow$ $< X_i^{22} >= 0$

↓

- Ambiguities in different slave-representations of $X^{\alpha\beta}$

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- A method is needed which uses only observables (gauge invariant quantities) related to $X^{\alpha\beta}$

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- Is there a technique for the composite object as a whole

↓

Answer - YES

⇓

- Baym-Kadanoff technique for $X^{\alpha\beta}$

↓

Baym-Kadanoff technique for $X^{\alpha\beta}$

↓

- Green's function in terms of sources

↓

- Self-energy in terms of vertex $\gamma_c(1,2;3)$ and $\gamma_s(1,2;3)$

↓

- **1/N expansion for** $U = \infty$

↓

- Quasiparticle spectrum

↓

Green's function

↓

- t-J Hamiltonian

↓

$$H_{tJ} = -t\sum_{ij\sigma} X_i^{\sigma 0} X_j^{0\sigma} + \frac{J}{2}\sum_{ij\sigma}(X_i^{\sigma\bar{\sigma}} X_j^{\bar{\sigma}\sigma} - X_i^{\sigma\sigma} X_j^{\bar{\sigma}\bar{\sigma}}) \tag{1}$$

↓

- External source generates higher Green's function

↓

$$H_{so}(\tau) = \sum_{i\sigma_1\sigma_2} U_i^{\sigma_1\sigma_2}(\tau) X_i^{\sigma_1\sigma_2}(\tau) \qquad (2)$$

$$\equiv \sum_i U_i^{\bar{\sigma}_1\bar{\sigma}_2}(\tau) X_i^{\bar{\sigma}_1\bar{\sigma}_2}(\tau)$$

↓

NOTATIONS

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- summation over bar spin indices $\bar{\sigma}$
- Notation ⇒ $1 = (i_1, \tau_1)$
- Integration over bar indices $\bar{1}$, $\bar{2}$,....

↓

$$\int d\tau H_{so}(\tau) \equiv U^{\bar{\sigma}_1\bar{\sigma}_2}(\bar{1}) X_1^{\bar{\sigma}_1\bar{\sigma}_2}(\bar{1})$$

$$= \sum_i \int_0^\beta d\tau U_i^{\bar{\sigma}_1\bar{\sigma}_2}(\tau) X_i^{\bar{\sigma}_1\bar{\sigma}_2}(\tau) \qquad (3)$$

↓

$$S = T_\tau e^{-U^{\bar{\sigma}_1\bar{\sigma}_2}(\bar{1}) X_1^{\bar{\sigma}_1\bar{\sigma}_2}(\bar{1})} \qquad (4)$$

- Matsubara (imaginary time) technique

↓

$$G^{\sigma_1\sigma_2}(1,2) = -\frac{< T_\tau(SX^{0\sigma_1}(1)X^{\sigma_2 0}(2)) \ >}{< T_\tau S >} \quad (5)$$

$$= - << X^{0\sigma_1}(1)X^{\sigma_2 0}(2) >>$$

↓

$$X^{0\sigma_1}(1) = e^{\tau H_{tJ}} X_{i_1}^{0\sigma_1} e^{-\tau H_{tJ}} \quad (6)$$

↓

- Definition

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$$G_{AB}(1,2) = - << A(1)B(2) >> \quad (7)$$

↓

- Equation of motion

↓

$$[-\frac{\partial}{\partial\tau_1}\delta^{\sigma_1\bar{\sigma}_1} - U^{\sigma_1\bar{\sigma}_1}(1)]\delta(1-\bar{1})G^{\bar{\sigma}_1\sigma_2}(\bar{1},2)$$

$$- t(1-\bar{1}) << X^{00}(1)X^{0\sigma_1}(\bar{1})X^{\sigma_2 0}(2) >>$$

$$- t(1-\bar{1}) \ \ll X^{\bar{\sigma}_1\sigma_1}(1)X^{0\bar{\sigma}_1}(\bar{1})X^{\sigma_2 0}(2) \gg$$

(8)

$$+ \text{"}J - terms\text{"} = \delta(1-2)Q^{\sigma_1\sigma_2}(1)$$

↓

- definition $t(1-2)$

↓

$$t(1-2) = t_{i_1 i_2}\delta(\tau_1 - \tau_2) \qquad \text{(8a)}$$

↓

- Hubbard spectral weight of the lower band $Q^{\sigma_1\sigma_2}$

↓

$$Q^{\sigma_1\sigma_2}(1) = \delta^{\sigma_1\sigma_2} \ll X^{00}(1) \gg + \ll X^{\sigma_1\sigma_2}(1) \gg$$

(9)

↓

- Baym-Kadanoff technique

⇒ obtaining higher Green's function by using lower ones

↓

- Price ⇒ functional equations

↓

- General rule

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$$<< X^{\sigma_1\sigma_2}(1)A(2)B(3) >>=$$

$$= \frac{\delta G_{AB}(2,3)}{\delta U^{\sigma_1\sigma_2}(1)} - G_{AB}(2,3) << X^{\sigma_1\sigma_2}(1) >> \qquad (10)$$

↓

- Three-points $(1, \bar{1}, 2)$ function in Eq.(8) can be expressed via $\delta G^{\sigma_1\sigma_2}(1,2)/ \delta U^{\sigma_{31}\sigma_4}(3)$

↓

"Quasiparticle" Green's function $g^{\sigma_1\sigma_2}(1,2)$

↓

$$g^{\sigma_1\sigma_2}(1,2) = G^{\sigma_1\bar{\sigma}_1}(1,2)Q^{-1,\bar{\sigma}_1\sigma_2}(1) \qquad (11)$$

↓

- Vertex function $\gamma^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(1,2;3)$

↓

$$\gamma^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(1,2;3) = -\frac{\delta g^{-1,\sigma_1\sigma_2}(1,2)}{\delta U^{\sigma_3\sigma_4}(3)} \qquad (12)$$

↓

- Some "tricks"

$\downarrow$

$$gg^{-1} = 1 \tag{13}$$

$\downarrow$

- functional derivative (schematically)

$$\frac{\delta g}{\delta U} g^{-1} + g \frac{\delta g^{-1}}{\delta U} = 0 \tag{14}$$

$\downarrow$

$$\frac{\delta g}{\delta U} = g \gamma g^{-1} \tag{15}$$

$\downarrow$

- Full expression of Eq.(15)

$\downarrow$

$$\frac{\delta g^{\sigma_1 \sigma_2}(1,2)}{\delta U^{\sigma_3 \sigma_4}(3)} = g^{\sigma_1 \bar{\sigma}_1}(1, \bar{1}) \gamma^{\bar{\sigma}_1 \bar{\sigma}_2}_{\sigma_3 \sigma_4}(\bar{1}, \bar{2}; 3) g^{\bar{\sigma}_2 \sigma_2}(\bar{2}, 2) \tag{16}$$

$\downarrow$

- Paramagnetic case

$\Downarrow$

$$g^{\sigma_1 \sigma_2}(1,2) = \delta^{\sigma_1 \sigma_2} g(1,2) \tag{17}$$

$$\Sigma^{\sigma_1\sigma_2}(1,2) = \delta^{\sigma_1\sigma_2}\Sigma(1,2) \qquad (18)$$

$$Q^{\sigma_1\sigma_2}(1) = \delta^{\sigma_1\sigma_2}Q(1) \qquad (19)$$

↓

- Note, $< A >$ means statistical average

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$$Q(1) =< X^{00} > + < X^{\sigma\sigma}(1) > \qquad (20)$$

↓

DYSON EQUATION FOR $g(1,2)$

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- For $U^{\sigma_3\sigma_4}(3) \to 0 \Rightarrow g(1,2) = g(1-2)$, etc.

↓

$$[G_{00}^{-1}(1-\bar{1}) - \Sigma(1-\bar{1})]g(\bar{1}-2) = \delta(1-2) \qquad (21)$$

$$G_{00}^{-1}(1-2) = -\frac{\partial}{\partial\tau_1}\delta(1-2) \qquad (22)$$

↓

$$\Sigma = \Sigma_H + \Sigma_\gamma + \Sigma_Q \qquad (23)$$

↓

$$\Sigma_H(1-2) = -Qt(1-2)$$

$$+\,\delta(1-2)J(1-\bar{1}) < X^{\sigma\sigma}(\bar{1}) > \qquad (24)$$

$\downarrow$

$$\Sigma_\gamma(1-2) = -t(1-\bar{1})g(\bar{1}-\bar{2})\gamma_c(2,\bar{2},1)$$

$$+\,t_J(1,\bar{3},\bar{1})g(\bar{1}-\bar{2})\gamma_s(\bar{2},2,\bar{3})] \qquad (25)$$

$\downarrow$

$$t_J(1,2,3) = \delta(1-2)t(1-3) - \delta(1-3)J(1-2)$$

$\downarrow$

- Charge vertex $\gamma_c(2,\bar{2},1)$

$\downarrow$

$$\gamma_c(2,\bar{2},1) \equiv \gamma^{\sigma\sigma}_{\bar{\sigma}\bar{\sigma}}(2,\bar{2},1) \qquad (26)$$

$\downarrow$

- Spin vertex $\gamma_s(2,\bar{2},1)$

$\downarrow$

$$\gamma_s(2,\bar{2},1) \equiv \gamma^{\bar{\sigma}\sigma}_{\bar{\sigma}\sigma}(2,\bar{2},1) \qquad (27)$$

$\downarrow$

- Summation over $\bar{\sigma}$ and integration over $\bar{1}$, $\bar{2}$, etc.

↓

- $\Sigma_Q(1-2)$ (of O(1/N) order) depends on $\delta Q/\delta U$

↓

- How to proceed further?
- How to calculate γ_c and γ_s in a controlable manner?

↓

1/N EXPANSION

↓

- N spin components
- New completness relation

↓

$$X_i^{00} + \sum_{\sigma=1}^{N} X_i^{\sigma\sigma} = \frac{N}{2} \qquad (28)$$

↓

- For $N = 2 \quad \Rightarrow \quad$ standard case
- Eq.(28) $\Rightarrow N/2$ spin ”projection” (spin ”up” or ”down”) can be occupied at the i-th lattice site!

↓

- For δ hole per lattice site one has

↓

$$\sum_{\sigma=1}^{N} < X_i^{\sigma\sigma} >= \frac{N}{2}(1-\delta) \qquad (29)$$

↓

- If Eq.(28) holds projection properties of $X^{\alpha\beta}$ **are lost**!

↓

$$X_i^{00} X_i^{00} \neq X_i^{00} \qquad (30)$$

$$X_i^{\sigma\sigma} \neq X_i^{\sigma 0} X_i^{0\sigma} \qquad (31)$$

↓

- Some important projections are **not lost**!

$$X_i^{\sigma\sigma} X_i^{\sigma\sigma} = X_i^{\sigma\sigma} \qquad (32a)$$

$$X_i^{00} X_i^{\sigma\sigma} = X_i^{\sigma 0} X_i^{0\sigma} - X_i^{\sigma\sigma} \qquad (32b)$$

- However one gets more $\Rightarrow$ small parameter (1/N) for large N!

↓

- **Scaling**: $t \Rightarrow t_0/N$ and $J \Rightarrow J_0/N$

$\downarrow$

$$Q = Nq_0 + Q_1 \tag{33}$$

$$Nq_0 =< X^{00} >= O(N) \tag{34}$$

$$Q_1 =< X^{\sigma\sigma} >= O(1) \tag{35}$$

$$g = g_0 + \frac{1}{N} g_1 + ... \tag{36}$$

$\downarrow$

$$\Sigma = \Sigma_0 + \frac{1}{N} \Sigma_1 + ... \tag{37}$$

$\downarrow$

$$\gamma_c = \gamma_{c,0} + \frac{1}{N} \gamma_{c,1} + ... \tag{38}$$

$$\gamma_s = N\delta(1-2)\delta(1-3) + \gamma_{s,1}(1,2;3) + ... \tag{39}$$

$\downarrow$

- Self-energy in the O(1) order Σ_0

$$\Sigma_0(1-2) = \delta(1-2)t_0(1-\bar{1})g_0(1-\bar{1}) - t_0(1-2)q_0 - J(1-2)g_0(1-2) \tag{40}$$

QuASIPARTICLE SPECTRUM IN O(1) ORDER

↓

- From Eq.(39) $\Rightarrow$ $\Sigma_0(\mathbf{k}, i\omega_n) = \Sigma_0(\mathbf{k})$

↓

$$\Sigma_0(\mathbf{k}) = \lambda - q_0 t_0(\mathbf{k}) - \frac{1}{N_c}\sum_{\mathbf{p}} J_0(\mathbf{k}+\mathbf{p}) n_F(\xi_{\mathbf{p}}) \tag{41}$$

$$\lambda = \frac{1}{N_c}\sum_{\mathbf{p}} t_0(\mathbf{p}) n_F(\xi_{\mathbf{p}}) \tag{42}$$

↓

$$g_0(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - (\Sigma_0(\mathbf{k}) - \mu)} \tag{43}$$

↓

- Quasiparticle spectrum: $g_0^{-1}(\mathbf{k}, i\omega_n) = 0$

↓

$$\xi_{\mathbf{k}} = \Sigma_0(\mathbf{k}) - \mu$$
$$= \lambda - q_0 t_0(\mathbf{k}) - \frac{1}{N_c}\sum_{\mathbf{p}} J_0(\mathbf{k}+\mathbf{p}) n_F(\xi_{\mathbf{p}}) \tag{44}$$

λ is a shift of the electronic level in order to accommodate $n = 1 - \delta$ particles under the Fermi surface

$\downarrow$

- Chemical potential μ

$$\frac{1-\delta}{2} = < X^{\sigma\sigma} >$$

$$= \frac{1}{N_c \beta} \sum_{i\omega_n \mathbf{k}} e^{0^+} g_0(\mathbf{k}, i\omega_n) = \sum_{\mathbf{k}} n_F(\xi_{\mathbf{k}}) \qquad (45)$$

$\downarrow$

IMPORTANT PROPERTIES OF O(1) ORDER

$\downarrow$

- Luttinger theorem is fulfilled i.e. under the Fermi surface are $1 - \delta$ particles as in the free-particle case!

$\downarrow$

- Large Fermi surface although small number ($\delta << 1$) of holes is present

$\Downarrow$

In the O(1) order $\Rightarrow$ Fermi liquid

DYNAMICAL CONDUCTIVITY $\sigma(\omega)$

$\downarrow$

$$\sigma^{xx}(\omega) = i\int_0^\infty dt e^{i(\omega+i\eta)} < [j_x(t), P_x] > \tag{46}$$

$\downarrow$

- Current density (N_c number of unit cells)

$\downarrow$

$$j_x(t) = \frac{1}{N_c}\dot{P}_x \tag{47}$$

$\downarrow$

- Polarization **P**

$$\mathbf{P} = \sum_i R_i X_i^{\bar{\sigma}\bar{\sigma}} = \sum_i R_i n_i \tag{48}$$

$\downarrow$

- Due to $[H, \sum n_i] = 0 \quad \Rightarrow$

$\downarrow$

$$< [P_x(t), P_x(0)] \quad >= \lim_{q\to 0} \frac{< n_{\mathbf{q}}(t) n_{-\mathbf{q}}(0) >}{q^2} \tag{49}$$

$\downarrow$

- Definition

$$\chi(\mathbf{q},\omega) = -\frac{i}{N_c}\int_0^\infty dt e^{i(\omega+i\eta)} < n_{\mathbf{q}}(t) n_{-\mathbf{q}}(0) > \tag{50}$$

$\downarrow$

$$\begin{aligned} \sigma^{xx}(\omega) &= -\int_0^\infty dt e^{i(\omega+i\eta)} < [j_x(t), P_x(0)] \ > \\ &= -\frac{\omega}{N_c}\int_0^\infty dt e^{i(\omega+i\eta)} < [P_x(0), P_x(t)] \ > \end{aligned} \tag{51}$$

$\downarrow$

O(N) ORDER OF $\sigma^{xx}(\omega)$

$\downarrow$

$$\sigma^{xx}(\omega) = \lim_{q\to 0}\frac{i\omega}{q^2}\chi(\mathbf{q},\omega) \approx \frac{1}{4} N\pi\delta\lambda\delta(\omega) \tag{52}$$

$\downarrow$

- In **strongly correlated** metals

$\Downarrow$

$\sigma^{xx}(\omega) \sim$ to the number of holes δ !

$\downarrow$

$$\sigma^{xx}(\omega) \sim \delta \tag{53}$$

$\downarrow$

- In **weakly correlated metals**

$$\Downarrow$$

$\sigma^{xx}(\omega) \sim$ number of carriers $(1-\delta)$!

$\downarrow$

$$\sigma^{xx}(\omega) \sim 1-\delta \tag{54}$$